



New inequalities for the exponential function[†]

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Abstract

In this paper, the authors present some monotonicity, convexity-concavity, log-convexity, log-concavity, subadditivity and superadditivity properties of certain combinations defined in terms of the function $u(x, y) = J_y(x) = e^x - (1 + x/y)^y$ which is a solution to the partial differential equation $(x + y)u_{yx} - yu_{yy} + u_x - u = 0$ for $x, y \in (0, \infty)$. By these properties, sharp double inequalities are obtained for the functions $J_y(x)$ and $R_{n+1}(x) \equiv J_{n+1}(x)^2 / [J_n(x)J_{n+2}(x)]$ for $x, y \in (0, \infty)$ and $n \in \mathbb{N}$, including the sharp Turán-type inequalities for $R_n(x)$ ($n \geq 2$), thus perfecting the related results proved recently by H. Alzer and solving his open problem on the best possible upper bound of $R_n(x)$.

Keywords: exponential function, monotonicity, convexity-concavity, log-convexity, log-concavity, subadditivity, superadditivity, inequalities.

MSC: 26D07, 39B62.

§1. Introduction

Throughout this paper, \mathbb{R} denotes the set of real numbers and \mathbb{N} denotes the set of positive integers,

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