Ejercicio 2 – Convocatoria Ordinaria 1 – Curso 2020/21

En un espacio vectorial V con base $B = \{e_1, e_2, e_3\}$, consideramos el subespacio vectorial

$$U = \{(x, y, z) \,/\, y = 0\}.$$

Se pide:

a) Calcular B_U una base y la dimensión de U.

b) ¿Pertenece al subespacio U el vector $v = e_1 - e_3$? En caso afirmativo, calcular las coordenadas de v en la base obtenida en el apartado anterior.

c) Consideremos en V el producto escalar cuya matriz de Gram respecto de B es:

$$G = \begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Se pide:

c.1. ¿Son perpendiculares los vectores e_1 y e_2? ¿Es el vector e_3 unitario? Razonar las respuestas.

c.2. ¿Forman los vectores $v = -e_2$ y $w = e_1 + e_2 - e_3$ un ángulo de 60º?

c.3. ¿Es B una base ortogonal? En caso negativo, calcular una base ortogonal.

a) $y = 0 \rightsquigarrow \left.\begin{array}{l} x = \alpha \\ y = 0 \\ z = \beta \end{array}\right\}$ Ec. paramétricas

$\alpha = 1$ y $\beta = 0 \rightsquigarrow (1, 0, 0)$

$\alpha = 0$ y $\beta = 1 \rightsquigarrow (0, 0, 1)$

$B_U = \{(1, 0, 0), (0, 0, 1)\}$ $\dim(U) = 2$.

b) $v = e_1 - e_3 \equiv (1, \underset{y}{0}, -1)_B \in U \iff \underline{\underline{y = 0}}$

$v \in U$

$(1, 0, -1) = x_1 (1, 0, 0) + x_2 (0, 0, 1)$

$= (x_1, 0, x_2) \implies \begin{array}{l} x_1 = 1 \\ 0 = 0 \\ x_2 = -1 \end{array}$

$v \equiv \underline{(1, -1)}_{B_U}$.

$$\langle x, y \rangle = (x_1, x_2, x_3) \begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

c.1) $e_1 \perp e_2 \iff \langle e_1, e_2 \rangle = 0$ ✓

$\|e_3\| = 1 \Rightarrow \sqrt{\langle e_3, e_3 \rangle} = 1 \iff \langle e_3, e_3 \rangle = 1$

c.2) $v = -e_2 \equiv (0,-1,0)$ $\quad \widehat{v, w} = 60^\circ$

$w = e_1 + e_2 - e_3 \equiv (1,1,-1)$

$$\cos(\alpha) = \frac{\langle v, w \rangle}{\|v\| . \|w\|} \qquad \cos(60^\circ) = \frac{1}{2}$$

$$\langle (0,-1,0), (1,1,-1) \rangle = (0,-1,0) \begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} =$$

$$= (0, -2, 0) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -2$$

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{(0,-1,0) \begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}} = \sqrt{(0,-2,0) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}} =$$

$$= \sqrt{2}$$

$$\|w\| = \sqrt{\langle w, w \rangle} = \sqrt{(1,1,-1) \begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} = \sqrt{(\sqrt{3}-1, 2, 0) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} =$$

$$= \sqrt{\sqrt{3} - 1}$$

$$\cos(\alpha) = \frac{-2}{\sqrt{2} . \sqrt{\sqrt{3}-1}} \neq \frac{1}{2} \rightsquigarrow \underline{\underline{\alpha \neq 60^\circ}}$$

c.3) $B=\{e_1,e_2,e_3\} \rightsquigarrow B'=\{u_1,u_2,u_3\}$

base ortogonal

$\langle u_i,u_j\rangle=0 \quad \forall i\neq j$

$$u_1=e_1$$

$$u_2=e_2-\frac{\langle e_2,u_1\rangle}{\langle u_1,u_1\rangle}\cdot u_1$$

$$u_3=e_3-\frac{\langle e_3,u_1\rangle}{\langle u_1,u_1\rangle}\cdot u_1-\frac{\langle e_3,u_2\rangle}{\langle u_2,u_2\rangle}\cdot u_2$$

$$e_1 = \underset{\substack{\| \\ 1}}{x_1}\, e_1 + \underset{\substack{\| \\ 0}}{x_2}\cdot e_2 + \underset{\substack{\| \\ 0}}{x_3}\cdot e_3 \rightsquigarrow e_1\equiv(1,0,0)_B$$

$$e_2\equiv(0,1,0)_B$$

$$e_3\equiv(0,0,1)_B$$

$$u_1=e_1=(1,0,0)$$

$$u_2=(0,1,0)-\underbrace{\cancel{\frac{\langle(0,1,0),(1,0,0)\rangle}{\langle(1,0,0),(1,0,0)\rangle}}}_{0}\cdot(1,0,0)=(0,1,0)=e_2$$

$$\langle(0,1,0),(1,0,0)\rangle=(0,1,0)\begin{pmatrix}\sqrt3&0&1\\0&2&0\\1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=(0,2,0)\begin{pmatrix}1\\0\\0\end{pmatrix}=0$$

$$u_3=(0,0,1)-\frac{\langle(0,0,1),(1,0,0)\rangle}{\langle(1,0,0),(1,0,0)\rangle}(1,0,0)-\underbrace{\cancel{\frac{\langle(0,0,1),(0,1,0)\rangle}{\langle(0,1,0),(0,1,0)\rangle}}}_{0}(0,1,0)$$

$$\langle(0,0,1),(1,0,0)\rangle=(0,0,1)\begin{pmatrix}\sqrt3&0&1\\0&2&0\\1&0&2\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=(1,0,2)\begin{pmatrix}1\\0\\0\end{pmatrix}=1$$

$$\langle(1,0,0),(1,0,0)\rangle=(1,0,0)\begin{pmatrix}\sqrt3&0&1\\0&2&0\\1&0&2\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=(\sqrt3,0,1)\begin{pmatrix}1\\0\\0\end{pmatrix}=\sqrt3$$

$$< (0,0,1),(0,1,0) > = (0,0,1)\begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$$

$$= (1,0,1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$u_3 = (0,0,1) - \frac{1}{\sqrt{3}}(1,0,0) = \left(-\frac{1}{\sqrt{3}}, 0, 1\right)$$

$$B' = \left\{ (1,0,0), (0,1,0), \left(-\frac{1}{\sqrt{3}}, 0, 1\right) \right\}$$