Ejercicio 4 – Ordinaria 1 – Curso 20/21

Sea f un endomorfismo en un espacio vectorial V con base $B = \{e_1, e_2, e_3\}$, del que sabemos que $e_1 - e_3 \in Ker(f)$ y que

$$f(e_1) = e_3, \qquad f(e_2 + 2e_3) = 2e_1 - e_2$$

a) ¿Es f diagonalizable? Razonar la respuesta.

b) Calcular, si es posible, base de vectores de $M_2(\mathbb{R})$ respecto de la cual, la matriz asociada a f sea diagonal.

c) Usar diagonalización para obtener la matriz asociada a f^4.

a)

$$\left.\begin{array}{l} e_1 - e_3 \in Ker(f) \Rightarrow f(e_1 - e_3) = 0 \\ \qquad\qquad\qquad\qquad\quad \shortparallel \\ \qquad\qquad\qquad\qquad f(e_1) - f(e_3) \end{array}\right\} \Rightarrow f(e_3) = f(e_1)$$

$$\underbrace{f(e_1) = e_3} \Rightarrow \underbrace{f(e_3) = e_3} = (0,0,1)$$

$$\underbrace{e_3 = 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3}$$

$$\left.\begin{array}{l} f(e_2 + 2e_3) = 2e_1 - e_2 \\ \qquad \shortparallel \\ f(e_2) + 2f(e_3) \end{array}\right\} \Rightarrow f(e_2) = 2e_1 - e_2 - 2e_3 = (2, -1, -2)$$

$$A = M_B(f) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 2 & 0 \\ 0 & -1-\lambda & 0 \\ 1 & -2 & 1-\lambda \end{vmatrix} = (-\lambda)(-1-\lambda)(1-\lambda) = 0$$

$\lambda = 0$ $\quad \lambda = -1$ $\quad \lambda = 1$

Todo los valores propios son distintos $\Rightarrow$ A es diag.

$\Downarrow$

f es diag.

b) $V_0 = \{ (x,y,z) \mid (A - 0I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{r} 2y = 0 \\ -y = 0 \\ x - 2y + z = 0 \end{array}\right\}$$

$y = 0$

$x + z = 0 \Rightarrow x = -z$

$$\left.\begin{array}{l} x = -\lambda \\ y = 0 \\ z = \lambda \end{array}\right\} \quad B_{V_0} = \{ (-1, 0, 1) \}$$

$V_{-1} = \{ (x,y,z) \mid (A - (-1)I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{r} x + 2y = 0 \\ x - 2y + 2z = 0 \end{array}\right|$$

$x = -2y$

$-4y + 2z = 0$

$\Downarrow$

$z = 2y$

$$\left.\begin{array}{l} x = -2\lambda \\ y = \lambda \\ z = 2\lambda \end{array}\right\} \Rightarrow B_{V_{-1}} = \{ (-2, 1, 2) \}$$

$V_1 = \{ (x,y,z) \mid (A - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{r} -x + 2y = 0 \\ -2y = 0 \\ \cancel{x - 2y = 0} \end{array}\right\}$$

$\Downarrow$

$y = 0 \quad z = \lambda$

$x = 0$

$B_{V_1} = \{ (0, 0, 1) \}$

$B' = \{(-1,0,1), (-2,1,2), (0,0,1)\}$

$$M_{B'}(f) = D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

c) A es diagonalizable $\Rightarrow$ $\exists D$ diagonal, $\exists P$ regular $\quad D = P^{-1}AP$

$$\Downarrow$$

$$PDP^{-1} = A$$

$$f^4 = f \circ f \circ f \circ f \Rightarrow f^4(x,y,z) = \underset{?}{A^4}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A^4 = P D^4 P^{-1} =$$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1} =$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}}$$

$$\mathrm{Adj}(P) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightsquigarrow \mathrm{Adj}(P)^t = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\underline{\underline{|P| = -1}} \quad \Longrightarrow \quad P^{-1} = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$f^4(x,y,z) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-2y,\ y,\ x+2y+z)$$